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ON THE STABILITY OF MOTION RELATIVE TO PART OF THE VARIABLES FOR CERTAIN NONLINEAR SYSTEMS

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The problem of motion stability relative to a part of the variables [1, 2] is examined. The method for solving this problem, proposed in [3] for linear stationary systems, is extended to solving nonlinear problems. Such an approach permits the obtaining of stability and instability criteria for motion relative to a part of the variables in the linear approximation in those cases when the well-known results in [4, 5] are inapplicable; it also yields a means for solving the problem posed here of absolute motion stability relative to a part of the variables for nonlinear controllable systems. Examples of nonlinear systems are cited, showing that the stability domain for separately specified coordinates can be wider than the stability domain for all the coordinates characterizing the system's state.

1. We consider the system of ordinary differential equations of perturbed motion

$$dx_i/dt = X_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n \tag{1.1}$$

We take up the question of the stability of the unperturbed motion $x_i = 0$ ($i = 1, \ldots, n$) relative to $x_1, \ldots, x_m$ ($m > 0$, $n = m + p$, $p > 0$). We denote these variables $y_i = x_i$ ($i = 1, \ldots, m$) and the ones remaining $z_j = x_{m+j}$ ($j = 1, \ldots, p$) [1, 2]. Let the functions X_i be power series in powers of y_i ($i = 1, \ldots, m$) and z_j ($j = 1, \ldots, p$), converging in the domains

$$|y_i| \leqslant h, \quad i = 1, \ldots, m; \quad |z_j| \leqslant H < \infty, \quad j = 1, \ldots, p \tag{1.2}$$

where h and H are some constants. Now the Eqs. (1.1) of perturbed motion are

$$\frac{dy_i}{dt} = \sum_{k=1}^{m} a_{ik} y_k + \sum_{l=1}^{p} b_{il} z_l + Y_i(y_1, \ldots, y_m, z_1, \ldots, z_p), \quad i = 1, \ldots, m \tag{1.3}$$

$$\frac{dz_j}{dt} = \sum_{k=1}^{m} c_{jk} y_k + \sum_{l=1}^{p} d_{jl} z_l + Z_j(y_1, \ldots, y_m, z_1, \ldots, z_p), \quad j = 1, \ldots, p$$

Here a_{ik}, b_{il}, c_{jk}, d_{jl} are constants, Y_i and Z_j are functions of the variables $y_1, \ldots, y_m$, $z_1, \ldots, z_p$, which in domains (1.2) are expanded into series in powers of these variables, where the expansions begin with terms of order no lower than the second. The variables $z_1, \ldots, z_p$ are always bounded; this assumption is the initial one in the investigation of system (1.3) in all the cases considered in Sects. 2 and 3.

We introduce the notation

$$A = \{a_{ik}\}, \quad B = \{b_{il}\}, \quad C = \{c_{jk}\}, \quad D = \{d_{jl}\}$$

Then the equations of linear approximation of system (1.3) are

$$\mathbf{y}^{\cdot} = A\mathbf{y} + B\mathbf{z}, \quad \mathbf{z}^{\cdot} = C\mathbf{y} + D\mathbf{z} \tag{1.4}$$
$$\mathbf{y} = (y_1, \ldots, y_m), \quad \mathbf{z} = (z_1, \ldots, z_p)$$

We consider the matrix $K = (B', D'B', \ldots, D'^{p-1}B')$, where B' and D' are the transposes of matrices B and D. Let the rank of matrix K equal N. It was shown in [3] that the question of the stability of the unperturbed motion of system (1.4) relative to variables $y_1, \ldots, y_m$ is equivalent to the same question for the specially constructed system

$$\mathbf{y}^{\cdot} = A\mathbf{y} + B_1\boldsymbol{\mu}, \quad \boldsymbol{\mu}^{\cdot} = C_1\mathbf{y} + D_1\boldsymbol{\mu}$$

of order $(m + N)$, $N \leqslant p$, where $\boldsymbol{\mu} = L\mathbf{z}$ is a matrix whose rows are the linearly-independent columns of matrix K; B_1, C_1, D_1 are constant matrices of appropriate dimensions. Hence we see that the question on the stability of the unperturbed motion of system (1.3) relative to a part of the variables $y_1, \ldots, y_m$ in the linear approximation can be considered in the following class of systems (we shall stay within the framework of the notation adopted and, as before, consider the variables $y_1, \ldots, y_m$ to be those relative to which the stability of the unperturbed motion is being studied):

$$\frac{dy_i}{dt} = \sum_{k=1}^{m} a_{ik}y_k + Y_i(y_1, \ldots, y_m, z_1, \ldots, z_p), \quad i = 1, \ldots, m \tag{1.5}$$

$$\frac{dz_j}{dt} = \sum_{k=1}^{m} c_{jk}y_k + \sum_{l=1}^{p} d_{jl}z_l + Z_j(y_1, \ldots, y_m, z_1, \ldots, z_p), \quad j = 1, \ldots, p$$

A criterion was obtained in [4] for the asymptotic stability of the unperturbed motion of system (1.5) relative to $y_1, \ldots, y_m$ in the linear approximation. An instability criterion was obtained in [5] and the critical case of one zero root was considered. The results mentioned were obtained under the following three constraints:

1) $\quad Y_i(0, \ldots, 0, z_1, \ldots, z_p) \equiv 0, \quad i = 1, \ldots, m$

2) $\quad |Y_i(y_1, \ldots, y_m, z_1, \ldots, z_p)| \leqslant \sum_{j=1}^{m} h_{ij}|y_j|, \quad i = 1, \ldots, m$

where h_{ij} are sufficiently small positive constants (under the assumptions made concerning functions Y_i the condition 2 assumes the terms linear in $y_1, \ldots, y_m$ are absent in functions Y_i);

3) the variables $z_1, \ldots, z_p$ of system (1.5) are always bounded, i.e., $|z_j| \leqslant H < \infty$, $j = 1, \ldots, p$.

In the present paper, for special cases of system (1.5), we have obtained stability and instability criteria for the unperturbed motion relative to variables $y_1, \ldots, y_m$ in the linear approximation, which do not presume constraints 1 and 2 on the functions $Y_i(y_1, \ldots, y_m, z_1, \ldots, z_p)$.

2. Let the Eqs. (1.3) of perturbed motion be

$$\frac{dy_i}{dt} = \sum_{k=1}^{m} a_{ik}y_k + \sum_{l=1}^{p} b_{il}z_l + Y_i(y_1, \ldots, y_m, z_1, \ldots, z_p), \qquad i = 1, \ldots, m \quad (2.1)$$
$$\frac{dz_j}{dt} = \sum_{l=1}^{p} d_{jl}z_l$$

We assume that

$$Y_i(0, \ldots, 0, z_1, \ldots, z_p) = Y_i^{\circ}(z_1, \ldots, z_p) = \quad (2.2)$$
$$U_2^{i}(z_1, \ldots, z_p) + \ldots + U_r^{i}(z_1, \ldots, z_p), \qquad i = 1, \ldots, m$$

where $U_l^i(z_1, \ldots, z_p)$ is a homogeneous form in variables $z_1, \ldots, z_p$ of order l, $l \leqslant r$, r is a finite number. From functions $Y_i(y_1, \ldots, y_m, z_1, \ldots, z_p)$ we pick out the terms linear in $y_1, \ldots, y_m$. We take it that

$$Y_i^*(y_1, \ldots, y_m, z_1, \ldots, z_p) = \sum_{j=1}^{m} y_j\overline{Y}_{ij}^*(z_1, \ldots, z_p) \quad (2.3)$$
$$(\overline{Y}_{ij}^*(z_1, \ldots, z_p) = \bar{U}_1^{ij}(z_1, \ldots, z_p) + \ldots + \bar{U}_s^{ij}(z_1, \ldots, z_p))$$

Here $\bar{U}_l^{ij}$ is a form of the same kind as $U_l^i(z_1, \ldots, z_p)$, $l \leqslant s$, s is a finite number. Thus, the functions $Y_i(y_1, \ldots, y_m, z_1, \ldots, z_p)$ in system (2.1) have the form

$$Y_i(y_1, \ldots, y_m, z_1, \ldots, z_p) = Y_i^{\circ}(z_1, \ldots, z_p) + \sum_{j=1}^{m} y_j\overline{Y}_{ij}^*(z_1, \ldots, z_p) + \quad (2.4)$$
$$Y_i^{**}(y_1, \ldots, y_m, z_1, \ldots, z_p), \qquad i = 1, \ldots, m$$

where $Y_i^{**}(y_1, \ldots, y_m, z_1, \ldots, z_p)$ satisfy conditions 1 and 2, while the functions $Y_i^{\circ}(z_1, \ldots, z_p)$ and $\overline{Y}_{ij}^*(z_1, \ldots, z_p)$ satisfy conditions (2.2) and (2.3).

Let us show that the question of the stability of the unperturbed motion of system (2.1) relative to $y_1, \ldots, y_m$ in the linear approximation can be reduced to that of the stability in the linear approximation of a specially chosen system for which the conditions is [4, 5] are fulfilled. To do this we take the following equations

$$\frac{dy_i}{dt} = \sum_{k=1}^{m} a_{ik}y_k + \sum_{l=1}^{p} b_{il}\dot{z}_l + Y_i^{\circ}(z_1, \ldots, z_p) + \quad (2.5)$$
$$\sum_{j=1}^{m} y_j\overline{Y}_{ij}^*(z_1, \ldots, z_p), \qquad i = 1, \ldots, m$$
$$\frac{dz_j}{dt} = \sum_{l=1}^{p} d_{jl}z_l, \quad j = 1, \ldots, p$$

as the first-approximation system for Eqs. (2.1). We introduce the new variables

$$\mu_i^{(1)} = \sum_{l=1}^{p} b_{il}z_l + U_2^{i}(z_1, \ldots, z_p) + \ldots + U_r^{i}(z_1, \ldots, z_p) \quad (2.6)$$

$$\mu_{ij}^{(2)} = \bar{U}_1^{(ij)}(z_1, \ldots, z_p) + \ldots + \bar{U}_s^{(ij)}(z_1, \ldots, z_p); \quad i, j = 1, \ldots, m$$

Since U_l^i and $\bar{U}_l^{ij}$ are l-th-order homogeneous forms in the variables $z_1, \ldots, z_p$, they can be written as

$$U_l^i = \sum_{\alpha_1+\ldots+\alpha_p=l} q_i^{\alpha_1\ldots\alpha_p} z_1^{\alpha_1} z_2^{\alpha_2} \ldots z_p^{\alpha_p}$$

$$\bar{U}_l^{ij} = \sum_{\alpha_1+\ldots+\alpha_p=l} \bar{q}_{ij}^{\alpha_1\ldots\alpha_p} z_1^{\alpha_1} z_2^{\alpha_2} \ldots z_p^{\alpha_p}$$

D e f i n i t i o n. The two collections of numbers $(\alpha_1, \ldots, \alpha_p)$ and $(\alpha'_1, \ldots, \alpha_p')$ are said to be distinct if $\alpha_i \neq \alpha_i'$ for even one i.

Let N_l be the number of distinct collections $(\alpha_1, \ldots, \alpha_p)$ such that $\alpha_1 + \ldots + \alpha_p = l$. Then with form U_l^i we associate the vector $(q_{il}^1, q_{il}^2, \ldots, q_{il}^{N_l})$, i.e., we associate the vector $\mathbf{Q}_i^{(1)}$ with the new variable $\mu_i^{(1)}$:

$$\mu_i^{(1)} \to \mathbf{Q}_i^{(1)} = (b_{i1}, \ldots, b_{ip}, q_{i1}^{11}, \ldots, q_{i1}^{1N_1}, \ldots, q_{ir}^{11}, \ldots, q_{ir}^{1N_r})$$

Analogously we associate the vector $\mathbf{Q}_{ij}^{(2)}$ with the new variable $\mu_{ij}^{(2)}$:

$$\mu_{ij}^{(2)} \to \mathbf{Q}_{ij}^{(2)} = (q_{ij1}^{21}, \ldots, q_{ij1}^{2N_1}, \ldots, q_{ijs}^{21}, \ldots, q_{ijs}^{2N_s})$$

We assume that vectors $\mathbf{Q}_i^{(1)}$ and $\mathbf{Q}_{ij}^{(2)}$ are linearly independent (otherwise, we consider those among them that are linearly independent). Two cases are possible with the thus-introduced new variables.

F i r s t c a s e. System (2.5) is reduced to the form

$$\frac{dy_i}{dt} = \sum_{k=1}^{m} a_{ik} y_k + \mu_i^{(1)} + \sum_{j=1}^{m} y_i \mu_{ij}^{(2)} \tag{2.7}$$

$$\frac{d\mu_j^{(1)}}{dt} = \sum_{l=1}^{m} L_{jl}^{(1)} \mu_l^{(1)} + \sum_{l,\varepsilon=1}^{m} \bar{L}_{jl\varepsilon}^{(1)} \mu_{l\varepsilon}^{(2)}$$

$$\frac{d\mu_{\nu\gamma}^{(2)}}{dt} = \sum_{l=1}^{m} L_{\nu\gamma l}^{(2)} \mu_l^{(1)} + \sum_{l,\varepsilon=1}^{m} \bar{L}_{\nu\gamma l\varepsilon}^{(2)} \mu_{l\varepsilon}^{(2)}, \qquad i, j, \nu, \gamma = 1, \ldots, m$$

In what follows (2.7) will be called the system of μ-form of the original system (2.5). It is obvious that the behavior of the variables characterizing system (2.7) completely describes the behavior of variables $y_1, \ldots, y_m$ of system (2.5).

S e c o n d c a s e. System (2.5) is not reduced to the system of μ-form, i.e. it appears as

$$\frac{dy_i}{dt} = \sum_{k=1}^{m} a_{ik} y_k + \mu_i^{(1)} + \sum_{j=1}^{m} y_j \mu_{ij}^{(2)} \tag{2.8}$$

$$\frac{d\mu_j^{(1)}}{dt} = U_1^{j*}(z_1, \ldots, z_p) + \ldots + U_r^{j*}(z_1, \ldots, z_p)$$

$$\frac{d\mu_{\nu\gamma}}{dt} = \overline{U}_2^{\nu\gamma*}(z_1, \ldots, z_p) + \ldots + \overline{U}_s^{\nu\gamma*}(z_1, \ldots, z_p); \quad i, j, \nu, \gamma = 1, \ldots, m$$

$$\frac{dz_\theta}{dt} = \sum_{l=1}^{p} d_{\theta l} z_p, \quad \theta = 1, \ldots, p$$

Then once more we introduce the new variables

$$\begin{aligned} &\overline{\mu}_i^{(1)} = U_1^{j*}(z_1, \ldots, z_p) + \ldots + U_r^{j*}(z_1, \ldots, z_p) \\ &\overline{\mu}_{\vartheta\gamma}^{(2)} = U_2^{\vartheta\gamma*}(z_1, \ldots, z_p) + \ldots + U_s^{\vartheta\gamma*}(z_1, \ldots z_p) \\ &i, \vartheta, \gamma = 1, \ldots, m \end{aligned} \tag{2.9}$$

where we choose only those of variables (2.9) whose corresponding vectors $\overline{\mathbf{Q}}_j^{(1)}$ and $\overline{\mathbf{Q}}_{\vartheta\gamma}^{(2)}$ cannot be represented in terms of $\mathbf{Q}_i^{(1)}$ and $\mathbf{Q}_{ij}^{(2)}$ $(i, j = 1, \ldots, m)$. It can be shown that by continuing this procedure, we can always come to a system of μ-form of system (2.5) at a finite step of repetition of the reasoning. Indeed, this can always be done by introducing variables (2.6), (2.9), ... in such number that the corresponding linearly-independent vectors $\mathbf{Q}_i^{(1)}$, $\overline{\mathbf{Q}}_j^{(1)}$ and $\mathbf{Q}_{ij}^{(2)}$, $\overline{\mathbf{Q}}_{\vartheta\gamma}^{(2)}$ form nonsingular square matrices.

We note that although the dimension of the μ-form equations can exceed that of the original system, the stability of the original system (2.5), relative to all variables will not, in general, follow from the stability of the unperturbed motion of the system of μ-form, as is true for linear stationary systems when the dimension of the system of μ-form equals that of the original system [3].

The two theorems that follow stem from the reasonings presented, as well as from the results in [4, 5].

T h e o r e m 1. If all the eigenvalues of the linear part of the μ-form equations of system (2.5) have negative real parts, then the unperturbed motion of system (2.1) is asymptotically stable relative to $y_1, \ldots, y_m$.

T h e o r e m 2. If among the eigenvalues of the linear part of the μ-form equations of system (2.5) there is even one with a positive real part, then the unperturbed motion of system (2.1) is unstable relative to $y_1, \ldots, y_m$.

3. Let the Eqs. (1.3) of perturbed motion be of the form

$$\frac{dy_i}{dt} = \sum_{k=1}^{m} a_{ik} y_k + \sum_{l=1}^{p} b_{il} z_l + Y_i(y_1, \ldots, y_m, z_1, \ldots, z_p), \quad i = 1, \ldots, m \tag{3.1}$$

$$\frac{dz_j}{dt} = \sum_{l=1}^{p} d_{jl} z_l + Z_j(y_1, \ldots, y_m, z_1, \ldots, z_p), \quad j = 1, \ldots, p$$

Let the following conditions be fulfilled:

a) the functions $Y_i(y_1, \ldots, y_m, z_1, \ldots, z_p)$ can be represented in form (2.2) —(2.4);

b) the functions $Z_j(y_1, \ldots, y_m, z_1, \ldots, z_p)$ satisfy conditions of type 1 and 2

$$Z_j(0, \ldots, 0, z_1, \ldots, z_p) \equiv 0$$

$$|Z_j(y_1, \ldots, y_m, z_1, \ldots, z_p)| \leqslant \sum_{i=1}^{m} \bar{h}_{ij} |y_j|, \qquad j = 1, \ldots, p$$

where $\bar{h}_{ij}$ are sufficiently small positive constants. In this case we can obtain criteria for system (3.1), similar to Theorems 1 and 2.

Example 1. Let the equations of perturbed motion be

$$\begin{aligned} &y^\cdot = -y + y\zeta + z_1\zeta^2 + Y(y, z_1, \ldots, z_4), \quad \zeta = 2z_1 + z_2 + z_3 + z_4 \\ &z_j^\cdot = \Sigma_j + Z_j(y, z_1, \ldots, z_4), \quad j = 1, \ldots, 4 \\ &(\Sigma_1 = -\zeta, \ \Sigma_2 = z_1 - z_2, \ \Sigma_3 = -4z_1 + z_2 - z_3 - z_4, \ \Sigma_4 = 5z_1 + z_2 + \\ &\quad 2z_3 + 2z_4) \end{aligned} \tag{3.2}$$

In the investigation of stability relative to all variables of the unperturbed motion of system (3.2) a critical case arises when the stability and the instability of the unperturbed motion is determined by the form of the nonlinear terms. Let us consider the question of the asymptotic stability of the unperturbed motion of system (3.2) relative to variable y. We note that the criterion in [4] is inapplicable here. We make use of the results in Sects. 2 and 3 of the present paper. We introduce the new variables: $\mu_1 = \zeta$, $\mu_2 = z_1\zeta^2$. System (3.2), after reduction to the system of μ-form as in Sect. 2, appears thus:

$$\begin{aligned} &y^\cdot = -y + y\mu_1 + \mu_2 + Y(y, z_1, \ldots, z_4) \\ &\mu_1^\cdot = -\mu_1 + Z_5(y, z_1, \ldots, z_4), \quad \mu_2^\cdot = \mu_3 + Z_6(y, z_1, \ldots, z_4) \\ &\mu_3^\cdot = -6\mu_2 - 5\mu_3 + Z_7(y, z_1, \ldots, z_4) \\ &z_j^\cdot = \Sigma_j + Z_j(y, z_1, \ldots, z_4) \\ &Z_5 = 2Z_1 + Z_2 + Z_3 + Z_4, \quad Z_6 = \zeta^2 Z_1 + 2z_1\zeta Z_5 \\ &Z_7 = \zeta^2(-2Z_1 + Z_3 + Z_4) + (-4z_1 + 2z_3 + 2z_4)\zeta Z_5 \end{aligned} \tag{3.3}$$

We assume the fulfilment of: conditions 1 and 2 by function $Y(y, z_1, \ldots, z_4)$, condition b) by function $Z_i(y, z_1, \ldots, z_4)$ $(i = 5, 6, 7)$, and condition 3. Then, according to [4], the unperturbed motion of system (3.3) is asymptotically stable relative to variables y, μ_1, μ_2, μ_3, and, hence, according to Sects. 2 and 3, the unperturbed motion of system (3.2) is asymptotically stable relative to variable y.

Note. Let us consider the case when the functions $Y_i^{**}(y_1, \ldots, y_m, z_1, \ldots, z_p)$ and $Z_j(y_1, \ldots, y_m, z_1, \ldots, z_p)$ occuring in the right-hand sides of system (3.1) are independent of variables $z_{r+1}, \ldots, z_p$, i.e., the functions Y_i^{**} and Z_j have the form

$$\begin{aligned} Y_i^{**} &= Y_i^{**}(y_1, \ldots, y_m, z_1, \ldots, z_r) \\ Z_j &= Z_j(y_1, \ldots, y_m, z_1, \ldots, z_r) \end{aligned} \tag{3.4}$$

In this case, theorems analogous to Theorems 1 and 2 can be obtained under the assumption that the variables $z_{r+1}, \ldots, z_p$ of system (3.1) are arbitrary, i.e., can be unbounded (but are z-continuable [2]). Indeed, let the following conditions be fulfilled, with due regard to (3.4), for the right-hand side of system (3.1):

A) $Y_i^{**}(0,\ldots,0, z_1,\ldots,z_r) \equiv 0, \quad Z_j(0,\ldots,0, z_1,\ldots,z_r) \equiv 0$;

B) functions (3.4) do not contain terms linear in $y_1,\ldots,y_m$;

C) the variables $z_1,\ldots,z_r$ of system (3.1) are always bounded, (the variables $z_{r+1},\ldots,z_p$ are arbitrary).

Under conditions A) — C) theorems analogous to Theorems 1 and 2 can be obtained by the method in [4, 5].

E x a m p l e 2. Let the equations of perturbed motion be

$$\begin{aligned} &y^\cdot = -y + z_2^2 z_3 + Y(y, z_1) \\ &z_1^\cdot = Z_1(z_1), \quad z_2^\cdot = -2z_2 + Z_2(y, z_1) \\ &z_3^\cdot = z_2 + 3z_3 + Z_3(y, z_1) \end{aligned} \tag{3.5}$$

We take it that variable z_1 of system (3.5) is always bounded. We consider the question of the asymptotic stability of the unperturbed motion of system (3.5) relative to variable y. After the reduction of system (3.5) to the system of μ-form as in Sects. 2 and 3, we obtain the following system of equations:

$$\begin{aligned} &y^\cdot = -y + \mu + Y(y, z_1) \\ &\mu^\cdot = \mu_1 + Z_4(y, z_1), \quad \mu_1^\cdot = -6\mu - 7\mu_1 + Z_5(y, z_1) \\ &z_1^\cdot = Z_1(z_1), \quad z_2^\cdot = -2z_2 + Z_2(y, z_1) \\ &z_3^\cdot = z_2 + 3z_3 + Z_3(y, z_1), \ \mu = z_2^2 z_3 \\ &(Z_4 = 2z_2z_3Y + z_2^2 Z_3, \ Z_5 = -2z_2z_3Y + 3z_2Y) \end{aligned}$$

We assume the fulfilment of: conditions 1 and 2 by function $Y(y, z_1)$ and of conditions A) and B) by functions $Z_4(y, z_1)$ and $Z_5(y, z_1)$. Then, according to [4] and to Sects. 2 and 3 of the present paper, the unperturbed motion of system (3.5) is asymptotically stable relative to y under the condition that variable z_1 is bounded.

4. Let Eqs. (1.3) of perturbed motion have the form

$$\begin{aligned} &\frac{dy_i}{dt} = Y_i(y_1,\ldots,y_m) + \sum_{l=1}^{p} b_{il} z_l, \quad i = 1,\ldots,m \\ &\frac{dz_j}{dt} = Z_j(y_1,\ldots,y_m) + \sum_{l=1}^{p} d_{jl} z_l, \quad j = 1,\ldots,p \end{aligned} \tag{4.1}$$

In contrast to Sects. 2 and 3 we do not require the boundedness of variables $z_1,\ldots, z_p$ of system (4.1) when investigating the latter. It is evident that the assumptions in [3] carry over completely to system (4.1). Krasovskii has investigated second- and third-order systems of form (4.1) in connection with the problem of motion stability in-the-large [6, 7]. We consider the system [6]

$$\begin{aligned} &y^\cdot = f_1(y) + b_{11}z_1 + b_{12}z_2 \\ &z_1^\cdot = f_2(y) + d_{11}z_1 + d_{12}z_2, \quad z_2^\cdot = f_3(y) + d_{21}z_1 + d_{22}z_2 \end{aligned} \tag{4.2}$$

We accept the fulfilment of the condition

$$\frac{b_{12}}{b_{11}} = \frac{b_{11}d_{12} - d_{11}b_{12}}{b_{12}d_{21} - b_{11}d_{22}} \tag{4.3}$$

Sufficient conditions were derived in [6], under which the unperturbed motion of system (4.2) is stable in-the-large. The domain in which these conditions hold is called domain Γ. Let us consider the question of the stability in-the-large of the solution $y = z_1 = z_2 = 0$ of system (4.2) relative to varialbe y and let us compare the stability domain obtained with domain Γ.

We introduce the new variable $\mu = b_{11}z_1 + b_{12}z_2$. Since condition (4.3) holds, in the new variables system (4.2) appears thus:

$$y^{\cdot} = f_1(y) + \mu, \quad \mu^{\cdot} = f_4(y) + k\mu \tag{4.4}$$

$$\left(f_4(y) = b_{11}f_2(y) + b_{12}f_3(y), \quad k = \frac{b_{12}d_{21} + b_{11}d_{11}}{b_{11}} = \frac{b_{11}d_{12} + b_{12}d_{22}}{b_{12}}\right)$$

and the behavior of variable y of system (4.2) is completely described by system (4.4). The domain in which the unperturbed motion of system (4.4) is stable in-the-large [7] is called domain Γ^*. It can be shown that domain Γ^* is wider than domain Γ, i.e., the domain of stability in-the-large of variable y of system (4.2) is wider than the domain of stability in-the-large of all the variables characterizing the state of this same system.

5. Let the equations of perturbed motion of a controllable system be

$$\frac{dy_i}{dt} = \sum_{k=1}^{m} a_{ik}y_k + \sum_{l=1}^{p} b_{il}z_l + h_i f(\sigma), \quad i = 1, \ldots, m \tag{5.1}$$

$$\frac{dz_j}{dt} = \sum_{k=1}^{m} c_{jk}y_k + \sum_{l=1}^{p} d_{jl}z_l + h_j f(\sigma), \quad j = 1, \ldots, p$$

$$\sigma = \beta_1 y_1 + \ldots + \beta_m y_m + \beta_{m+1} z_1 + \ldots + \beta_n z_p$$

where a_{ik}, b_{il}, c_{jk}, d_{jl}, h_i, h_j, β_i are constants and $f(\sigma)$ is a continuous function satisfying the condition

$$\sigma f(\sigma) > 0 \quad \text{when} \quad \delta \neq 0 \tag{5.2}$$

In the investigation of system (5.1) we do not require the boundedness of its variables $z_1, \ldots, z_p$. Let us consider the problem of the absolute stability of the unperturbed motion of system (5.1) relative to variables $y_1, \ldots, y_m$. This problem generalizes the well-known Lur'e problem [8].

D e f i n i t i o n. The unperturbed motion of system (5.1) is said to be absolutely stable relative to $y_1, \ldots, y_m$ if it is stable relative to $y_1, \ldots, y_m$ under any initial deviations and for any choice of function $f(\sigma)$ satisfying condition (5.2).

Let us show that the problem posed can be reduced to the problem of absolute stability of a specially chosen system of the same form relative to all variables, where the latter system's dimension can be less than that of the original system. To do this, following [3] we introduce the new variables

$$\mu_i = b_{i1}z_1 + \ldots + b_{ip}z_p, \quad i = 1, \ldots, m \tag{5.3}$$

$$\eta = \beta_{m+1}z_1 + \ldots + \beta_n z_p$$

(we assume that variables (5.3) are linearly independent; otherwise, from (5.3) we choose the linearly-independent ones).

Two cases are possible when such new variables are introduced. In the first, system (5.2) is reduced to

$$\frac{dy_i}{dt}=\sum_{k=1}^{m}a_{ik}y_k+\mu_i+h_if(\sigma) \tag{5.4}$$

$$\frac{d\mu_j}{dt}=\sum_{k=1}^{m}c_{jk}{}^*y_k+\sum_{l=1}^{m}e_{jl}\mu_l+e_j{}^*\eta+h_j{}^*f(\sigma),\quad i,j=1,\ldots,m$$

$$\frac{d\eta}{dt}\sum_{k=1}^{m}\bar{c}_k{}^*y_k+\sum_{l=1}^{m}\bar{e}_l\mu_l+\bar{e}^*\eta+\bar{h}^*f(\sigma)$$

$$\sigma=\beta_1y_1+\ldots+\beta_my_m+\eta$$

i.e., the behavior of the variables characterizing the state of system (5.4) determines completely the behavior of variables $y_1,\ldots,y_m$ of system (5.1). Systems of form (5.4) are called systems of μ-form of the original system (5.1). In the second case, when system (5.1) does not, after the introduction of variables (5.3), reduce to the μ-form, by the scheme in [3] we can show that system (5.1) can always be brought to the μ-form at a finite stage of repeating the procedure of introducing new variables. The dimension of the system of μ-form does not exceed that of the original system (5.1); to be precise, the following is valid.

L e m m a. In order that the dimension of the system of μ-form of system (5.1) equal N, it is necessary and sufficient that the rank of the matrix $K=(B, DB,\ldots, D^{p-1}B)$ equal $N-m$ (here $B=\{b_{il},\beta_l\}$, and $D=\{d_{lj}\}$ are matrices of appropriate dimensions).

Thus we obtain the following result.

T h e o r e m 3. For the absolute stability of the unperturbed motion of system (5.1) relative to $y_1,\ldots,y_m$, it is sufficient that the system of μ-form be absolutely stable with respect to all variables. When the rank of matrix K equals p, the problem being analyzed is equivalent to the Lur'e problem.

Let us present an example of the effective use of the method indicated. This example shows it is possible to have automatic control systems that are not absolutely stable with respect to all variables, but can be absolutely stable with respect to a part of the variables.

We consider the case when the matrix of the linear part of equation system (5.1) has two zero roots, i.e., Eqs. (5.1) are

$$\frac{dy_i}{dt}=\sum_{k=1}^{m}a_{ik}y_k+h_if(\sigma),\quad i=1,\ldots,m \tag{5.5}$$

$$dz_1/dt=\gamma_1f(\sigma),\quad dz_2/dt=\gamma_2f(\sigma);\quad \sigma=\mathbf{c}'\mathbf{y}+\beta_1z_1+\beta_2z_2$$

where γ_1 and γ_2 are constants, $\mathbf{c}'$ is a constant vector. System (5.5) has a nonzero equilibrium position; therefore, its unperturbed motion cannot be absolutely stable in all variables [9]. Let us consider the problem of the absolute stability of the unperturbed motion of system (5.5) relative to $y_1, \ldots, y_m$. For this we introduce a new variable: $\gamma\mu = \beta_1 z_1 + \beta_2 z_2$, where $\gamma < 0$ is a constant. The system of μ-form appears as

$$\frac{dy_i}{dt} = \sum_{k=1}^{m} a_{ik} y_k + h_i f(\sigma), \quad i = 1, \ldots, m \tag{5.6}$$

$$d\mu/dt = \Gamma f(\sigma); \quad \sigma = \mathbf{c}'\mathbf{y} + \gamma\mu, \quad \gamma < 0; \quad \Gamma = 1/\gamma\,(\beta_1 z_1 + \beta_2 z_2)$$

The well known absolute stability conditions for system (5.6) (see [9]) are the stability conditions for system (5.5) relative to $y_1, \ldots, y_m$.

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